ASSIGNMENT 3
UNIFORM DISTRIBUTION THEORY 2021
DUE DATE: APRIL 21, 2021

Review of the basic theory of continued fractions: For an irrational real number $x \in \mathbb{R}\backslash\mathbb{Q}$, the continued fraction expansion is defined by

$$x = [a_0; a_1, a_2, a_3, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

where a_j are integers (the "partial quotients"), $a_1, a_2, a_3, \dots \geq 1$, defined by writing

$$x = a_0 + \frac{1}{x_1}, \quad x_1 = a_1 + \frac{1}{x_2}, \dots$$

with $x_n > 1$, so that for $n \geq 1$, $a_n = \lfloor x_n \rfloor$, $x_{n+1} = 1/\{x_n\}$. In particular

$$x = [a_0; a_1, a_2, \dots, a_{n-1}, x_n].$$

We also define recursively $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = a_0$, $q_0 = 1$, and for $n \geq 0$

$$p_{n+1} = a_{n+1}p_n + p_{n-1}, \; q_{n+1} = a_{n+1}q_n + q_{n-1}.$$

Exercise 1. a) Show that $p_n q_{n-1} - p_{n-1}q_n = (-1)^{n-1}$.

b) Show that

$$[a_0; a_1, a_2, a_3, \dots, a_n] = \frac{p_n}{q_n} \quad \text{and} \quad x = \frac{x_{n+1}p_n + p_{n-1}}{x_{n+1}q_n + q_{n-1}}.$$

Exercise 2. a) Show

$$x - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n(x_{n+1}q_n + q_{n-1})}$$

b) Show that

$$|x - \frac{p_n}{q_n}| \leq \frac{1}{a_{n+1}q_n^2}.$$

Hence the "partial convergents" p_n/q_n give good rational approximations to x: $|x - p_n/q_n| < 1/q_n^2$.

Exercise 3. Let x be an algebraic number of degree $d \geq 2$, with partial convergents p_n/q_n. Show that $\log q_n \ll d^n$.

Hint:Use Liouville's theorem and Exercise 2.